A Terse Introduction to the Mathematical Theory of Wavelets

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Abstract

This document is a write up of my course notes on wavelets taught by Pencho Petrushev. Sections 1-4 is the material provided from the course. Section 5 contains some of my more recent research into the evaluation of computing wavelet decompositions and reconstructions in practice.

Contents

1 Introduction and The Multiresolution Analysis (MRA)

The most foundational component to wavelet theory is the definition of MRA.

Definition 1 (MRA)**.** *$V_n \subset L_2(\mathbb{R})$, $n \in \mathbb{Z}, V_n$ is a subspace.*

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$
2. $span\,(\cup_n V_n) = L_2(\mathbb{R})$
3. $\cap_n V_n = \{0\}$
4. $f(x) \in V_j \iff f(2^{-j}x) \in V_0$
5. $f \in V_0 \iff f(x-m) \in V_0, \forall m$
6. $\exists \phi \in V_0\, s.t.\, \{\phi(x-n)\}_{n\in\mathbb{Z}}$ *is an O.N.B. for* V_0. ϕ *is the scaling function.*

Remark 1. $\{2^j\phi(2^{j/2}x-k)\}_{k\in\mathbb{Z}}$ *is an O.N.B. for* V_j.

1.1 Haar Basis: The simplest wavelet basis

Before diving into the rigorous mathematical theory, we provide the definition of the simplest wavelet basis, the Haar basis.

- Haar scaling function: $\phi(x) = \mathbb{1}_{[0,1)}(x)$.
- Haar Mother wavelet: $\psi(x) = \begin{cases} 1 & \text{if } x \in [0,1/2) \\ -1 & \text{if } x \in [1/2,1) \\ 0 & \text{if O.W.} \end{cases}$
- $\phi_{jk} = 2^{j/2}\phi(2^jx-k)$
- $\psi_{jk} = 2^{j/2}\psi(2^jx-k)$

Let $f \in V_n$ where

$$V_n := \left\{ f(x) = \sum_{k=0}^{2^n-1} p_k \mathbb{1}_{[k/2^n,(k+1)/2^n)} \right\},$$

then

$$f(x) = \sum_{k=0}^{2^n-1} c_k^n \phi_{nk}(x),$$

where $c_k^n = \langle f, \phi_{nk}\rangle$. Moreover, the wavelet basis for V_n is given by

$$\phi_{00} \cup \{\{\psi_{jk}\}_{k=0}^{2^j-1}\}_{j=0}^{n-1}.$$

Hence, for $f \in V_n$,

$$f(x) = c_0^0\phi_{00}(x) + \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} d_k^j \psi_{jk}(x),$$

where $d_k^j = \langle f, \psi_{jk}\rangle$. d_k^j are the *detail* coefficients and c_k^j are the *approximation* coefficients. To obtain a basis for $L_2[0,1]$ we take the limit as $n \to \infty$. In practice in the digital domain, n is of course finite.

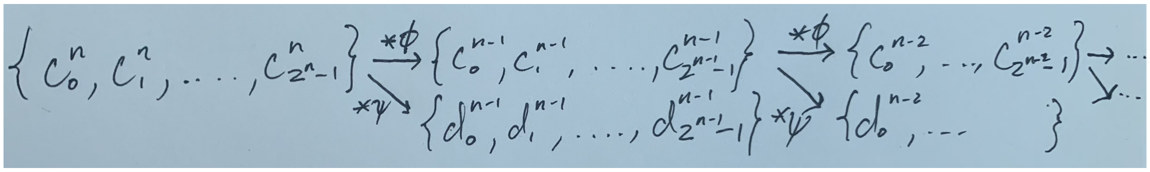

Figure 1: Diagram for fast calculation of the wavelet coefficients.

1.2 Quickly calculating the coefficients (illustrated with Haar)

First note

$$\phi_{jk} = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k}(x) + \phi_{j+1,2k+1}(x)\right),$$

hence

$$\begin{aligned} c_k^j &= \frac{1}{\sqrt{2}} \int_0^1 f(x) \left(\phi_{j+1,2k}(x) + \phi_{j+1,2k+1}(x)\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2}} \left(c_{2k}^{j+1} + c_{2k+1}^{j+1}\right) \end{aligned} \tag{1}$$

Similarly noting

$$\psi_{jk} = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k} - \phi_{j+1,2k+1}\right),$$

hence

$$d_k^j = \frac{1}{\sqrt{2}} \left(c_{2k}^{j+1} - c_{2k+1}^{j+1}\right)$$

Hence, the larger element coefficients from the basis (lower frequency elements) d_k^j and c_k^j are computed from the smaller elements (higher frequency) coefficients one level up. Moroever, they may be evaluated by high pass filtering and low pass filtering (with ψ and ϕ), respectively, followed by a downsampling (keeping every other coefficient, since there are half as many as the levels go down). See Figure 1 for an illustration.

2 Important Preliminary Results

Let H be a Hilbert Space. We say $\{x_n\}_{n\in\mathbb{Z}} \subset H$ is a Riesz sequence if

$$c_1 \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2} \leq \left\|\sum_{n\in\mathbb{Z}} a_n x_n\right\|_H \leq c_2 \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2}, \tag{2}$$

$\forall (a_n)_n$ with $c_1, c_2 > 0$. Also, $\{x_n\}_n$ is called a Riesz basis if $\mathrm{span}\{x_n\}_{n\in\mathbb{Z}} = H$.

Proposition 1. *Let $\phi \in L_2(\mathbb{R})$. Then*

(i)

$$a \left(\sum_n |a_n|^2\right)^{1/2} \leq \left\|\sum_{n\in\mathbb{Z}} a_n \phi(x-n)\right\|_2 \leq A \left(\sum_n |a_n|^2\right)^{1/2}, \ \forall \{a_n\}_{n\in\mathbb{Z}} \in \ell_2$$

$\Longleftrightarrow$

(ii) For almost all $\xi \in [0, 2\pi)$,

$$\frac{a^2}{2\pi} \le \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2 \le \frac{A^2}{2\pi}.$$

The proof is provided in the appendix.

Corollary 1. $\phi \in L_2(\mathbb{R})$. $\{\phi(x-n)\}_{n\in\mathbb{Z}}$ *is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2 = 1/2\pi$$

almost everywhere.

If $g \in \text{span}\{\phi(x-n)\}_n$, then $g = \sum_{n\in\mathbb{Z}} a_n \phi(x-n)$ and

$$\hat{g}(\xi) = \underbrace{\left(\sum_{n\in\mathbb{Z}} a_n e^{-in\xi}\right)}_{h(\xi)} \hat{\phi}(\xi). \tag{3}$$

Moreover

$$\|g\|_2^2 = \int_0^{2\pi} |h(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)| \, \mathrm{d}\xi \le \frac{A^2}{2\pi} \int_0^{2\pi} |h(\xi)|^2 \, \mathrm{d}\xi,$$

hence

$$A^{-1}\|g\|_2 \le \left(\frac{1}{2\pi}\int_0^{2\pi} |h(\xi)|^2 \, \mathrm{d}\xi\right) \le a^{-1}\|g\|_2$$

Proposition 2. *Under the conditions of the previous proposition 1, there exists* $\phi_1 \in$ *span*$\{\phi(x-n)\}_n$ *such that* $\{\phi_1(x-n)\}_{n\in\mathbb{Z}}$ *forms an orthonormal basis for span*$\{\phi(x-n)\}_n$.

proof of Proposition 2. $\hat{\phi}_1(\xi) = h(\xi)\hat{\phi}(\xi)$, like above in (3), and

$$\frac{1}{2\pi} = \sum_{k\in\mathbb{Z}} \underbrace{|h(\xi + 2\pi k)|^2}_{2\pi\text{periodic}} |\hat{\phi}(\xi + 2\pi k)|^2 = |h(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2$$

by the previous corollary. Choose h to be real and set

$$\hat{\phi}_1(\xi) = \frac{\hat{\phi}(\xi)}{\sqrt{2\pi}\left(\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2\right)^{1/2}}.$$

Then $\phi_1 \in \text{span}\{\phi(x-n)\}_n$ since it takes the form (3) and

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}_1(\xi + 2\pi k)|^2 = \frac{1}{2\pi},$$

which by Corollary 1 implies it form an orthonormal basis. $\square$

3 The Scaling Equation

Let $\phi(x) \in V_0$. Then

$$\begin{aligned}
\phi(x/2) &= \sum_{n\in\mathbb{Z}} a_n \phi(x-n) \\
\phi(x) &= \sum_{n\in\mathbb{Z}} a_n \phi(2x-n) \\
\widehat{\phi(x/2)}(\xi) &= 2\hat{\phi}(2\xi) \\
\hat{\phi}(2\xi) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{-in\xi}\hat{\phi}(\xi) = m(\xi)\hat{\phi}(\xi)
\end{aligned} \tag{4}$$

or equivalently

$$\hat{\phi}(\xi) = m(\xi/2)\hat{\phi}(\xi/2), \tag{5}$$

where $m(\xi)$ is 2π periodic.

$$\sqrt{2} = \|\phi(x/2)\|_2 = \left(\sum_n |a_n|^2\right)^{1/2} \tag{6}$$

Lemma 1.

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1. \tag{7}$$

proof of Lemma 1.

$$\begin{aligned}
1/2\pi &= \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi k)|^2 \\
&= \sum_{k\in\mathbb{Z}} |m(\xi/2+\pi k)|^2 |\hat{\phi}(\xi/2+\pi k)|^2 \\
&= \sum_{k\in\mathbb{Z}} |m(\xi/2+2\pi k)|^2 |\hat{\phi}(\xi/2+2\pi k)|^2 + |m(\xi/2+2\pi k+\pi)|^2 |\hat{\phi}(\xi/2+2\pi k+\pi)|^2 \\
&= |m(\xi/2)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi/2+2\pi k)|^2 + |m(\xi/2+\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi/2+2\pi k+\pi)|^2 \\
&= |m(\xi/2)|^2/2\pi + |m(\xi/2+\pi)|^2/2\pi.
\end{aligned} \tag{8}$$

□

Theorem 1. *Let $\phi \in L_2(\mathbb{R})$. Suppose*

1.
$$a\left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2} \le \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|_2 \le A\left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2}, \forall\{a_n\}_n$$

2. $\phi(x/2) = \sum_{n\in\mathbb{Z}} a_n\phi(x-n)$ *in the L_2 sense.*

3. $\hat{\phi}(\xi)$ *is continuous at $\xi = 0$ and $\hat{\phi}(0) \neq 0 \iff \int_{\mathbb{R}} \phi\, dx \neq 0$.*

Let $V_j = span\{\phi(2^j x - k)\}_{k\in\ Z}$, $\forall j \in \mathbb{Z}$. Then $\{V_j\}_{j\in\ Z}$ forms a multiresolutional analysis (MRA).

This result is basically just a culmination of some of the previous results, along with a few claims that need to be proven. It is left as an exercise to the reader.

4 Construction of Wavelets

Recall

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$$

Denote

$$W_j := V_{j+1} \ominus V_j \iff V_j \oplus W_j = V_{j+1}$$

Then W_j is the set of all $f \in V_{j+1}$ such that $f \perp V_j$. Then

$$V_{j+1} = \oplus_{v=-\infty}^{j} W_v$$

and

$$L_2(\mathbb{R}) - \oplus_{j\in\mathbb{Z}} W_j.$$

Now there exists $\psi \in W_0$ such that $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ is an othonormal basis for W_0, and then $\{\psi_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j, where $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. Finally then $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{R})$.

4.1 Complete Characterization in Fourier Domain

Our goal is to find $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ that is an orthonormal basis for W_0. Return to the scaling equation:

$$\phi(x/2) = \sum_{n\in\mathbb{Z}} a_n \phi(x-n) \iff \hat{\phi}(2\xi) = \underbrace{m(\xi)}_{2\pi\text{periodic}} \hat{\phi}(\xi),$$

where

$$m(\xi) = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{-in\xi}.$$

Hence $\hat{\phi}(\xi) = m(\xi/2)\hat{\phi}(\xi/2)$. Also recall

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2 = \frac{1}{2\pi},$$

and

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$$

Claim:

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} = 1/\sqrt{2}. \tag{9}$$

proof of (9).

$$\int_{\mathbb{R}} |\phi(x/2)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\phi(y)|^2 2 \,\mathrm{d}y = 2,$$

hence $\sqrt{2} = \|\phi(x/2)\|_2 = \|2\hat{\phi}(2\xi)\|_2 = 2\|m(\xi)\hat{\phi}(\xi)\|_2$, and finally

$$\begin{aligned}
\sqrt{2}/2 &= \|m(\xi)\hat{\phi}(\xi)\|_2 \\
&= \left(\int_{\mathbb{R}} |m(\xi)|^2 |\hat{\phi}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} \\
&= \left(\int_0^{2\pi} \sum_{k\in\mathbb{Z}} |m(\xi+2\pi k)|^2 |\hat{\phi}(\xi+2\pi k)|^2 \,\mathrm{d}\xi\right)^{1/2} \\
&= \left(\int_0^{2\pi} |m(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi k)|^2 \,\mathrm{d}\xi\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2}
\end{aligned} \tag{10}$$

□

Using similar argument to the above proof

$$g \in V_0 \iff \hat{g}(\xi) = m_g(\xi)\hat{\phi}(\xi),$$

moreover

$$\|g\|_2 = \left(\frac{1}{2\pi}\int_{\mathbb{R}} |m_g(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2}.$$

We have

$$f \in V_1 \iff f(x) = \sqrt{2}g(2x), \text{ for some } g \in V_0,$$

and if follows that

- $\|f\|_2 = \|g\|_2$.
- $\hat{f}(\xi) = 1/\sqrt{2}\hat{g}(\xi/2)$
- $\hat{f}(\xi) = m_f(\xi/2)\hat{\phi}(\xi/2)$
- $1/\sqrt{2}\|f\|_2 = \left(1/2\pi \int_0^{2\pi} |m_f(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2}$.

The first two bullet points follow trivially. The third follows from

$$\begin{aligned} f(x) &= \sum_n a_n \phi(2x-n) \\ \Rightarrow \hat{f}(\xi) &= \sum_n a_n \int_{\mathbb{R}} \phi(2x-n) e^{-ix\xi}\, \mathrm{d}x \\ &= \sum_n a_n \int_{\mathbb{R}} \phi(y) e^{-i\xi/2(y+n)}/2\, \mathrm{d}y \\ &= \frac{1}{2}\sum_n a_n e^{-i\xi/2n} \hat{\phi}(\xi/2) \\ &= m_f(\xi/2)\hat{\phi}(\xi/2). \end{aligned} \tag{11}$$

The last line follows by similar calculations performed before by manipulating the above equation.

Proposition 3. *If $u, v, x, y \in \mathbb{C}$, and (u, v) is orthogonal to (x, y), then $(u, v) = \alpha(\overline{y}, -\overline{x})$ for some $\alpha \in \mathbb{C}$.*

The proof is provided in the appendix.

Proposition 4. *$f \in W_0$ if and only if*

$$\hat{f}(\xi) = e^{i\xi/2} V(\xi) \overline{m_\phi(\xi/2+\pi)} \hat{\phi}(\xi/2+\pi),$$

almost everywhere, where $V(\xi)$ is 2π-periodic. Moreover,

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |V(\xi)|^2 \, d\xi \right)^{1/2}.$$

The proof is provided in the appendix.

Lemma 2. *$f \in W_0$, then $\{f(x-n)\}_n$ is an orthonormal basis for W_0 if and only if $|V(\xi)| = 1$ a.e.*

proof of Lemma 2. Recall $\{f(x-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\sum_{k \in \ Z} |\hat{f}(\xi + 2\pi k)|^2 = (2\pi)^{-1}.$$

Then

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |\hat{f}(\xi+2\pi k)|^2 &= \sum_{k\in\mathbb{Z}} |V(\xi)|^2 |m_\phi(\xi/2+\pi k)|^2 |\hat{\phi}(\xi/2+\pi k)|^2 \\ &= |V(\xi)|^2 \Bigg[\sum_{k\in\mathbb{Z}} |m_\phi(\xi/2+2\pi k)|^2 |\hat{\phi}(\xi/2+2\pi k)|^2 \\ &\quad + \sum_{k\in\mathbb{Z}} |m_\phi(\xi/2+\pi+2\pi k)|^2 |\hat{\phi}(\xi/2+\pi+2\pi k)|^2 \Bigg] \\ &= |V(\xi)|^2/2\pi \end{aligned} \tag{12}$$

□

Theorem 2. *ψ is a mother wavelet and $\{\psi(x-n)\}_n$ is an orthonormal basis for W_0 if and only if*

$$\hat{\psi}(\xi) = e^{i\xi/2} V(\xi) \overline{m_\phi(\xi/2+\pi)} \hat{\phi}(\xi/2), \tag{13}$$

where $|V(\xi)| = 1$ a.e. and $V(\xi)$ is 2π-periodic.

4.2 Back To The Real Space

The characterization of a wavelet basis is essentially provided in the Fourier domain by Theorem 2. Choose $V(\xi) = 1$ and see what happens. In this case

$$\begin{aligned} \hat{\psi}(\xi) &= \frac{1}{2} e^{i\xi/2} \sum_{n\in\mathbb{Z}} \overline{a_n} e^{in(\xi/2+\pi)} \hat{\phi}(\xi/2) \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}} (-1)^n \overline{a_n} e^{i(n+1)\xi/2} \hat{\phi}(\xi/2) \end{aligned} \tag{14}$$

Then

$$\psi(x) = \sum_{n\in\mathbb{Z}} \overline{a_n} (-1)^n \phi(2x+n+1),$$

where

$$a_n = \int \phi(x/2) \overline{\phi(x-n)} \mathrm{d}x.$$

Other constructions of wavelets are formed by simply using (13) and choosing different functions for V such that $|V(\xi)| = 1$.

5 Fast Wavelet Transform

This section loosely provides the foundation and ideas for computing fast wavelet transforms in practice.

5.1 Basics

Let the original signal be denoted by s. The orthonormal wavelet representation with ℓ levels is written as

$$s(x) = \sum_{j=1}^{\ell} \sum_k d_{j,k} \psi_{j,k}(x) + \sum_k a_k \phi_k(x) \tag{15}$$

The index j represents the scale (level) and k the shift. Contrary to the notation used earlier, larger values of the level j correspond to wider basis functions (lower frequency components). The $d_{j,k}$ are the detail coefficients and the a_k are the approximation coefficients for the scaling function ϕ. ψ are the wavelets, where now the notation is

$$\psi_{j,k}(x) = c\psi(2^{-j+1}x - k),$$

where c is a normalization. For a signal of length 2^N there are 2^{N-j} basis elements at the jth level, and $2^{N-\ell}$ basis scaling functions. As a sanity check, one may confirm adding all of those numbers together gives you 2^N basis functions.

5.2 Fast Decomposition through layered filtering

Since (15) is an orthonormal representation

$$d_{j,k} = \langle s, \psi_{j,k} \rangle.$$

To perform these operations fast in practice, we make use of relationships between the levels and convolution operations, with ψ and ϕ. The functions ψ and ϕ without a subscript denote these basis functions in their form with smallest support (highest frequency). In general, convolution with ψ and ϕ act as HPF and LPF respectively. The detail coefficients $d_{j,k}$ at level j are extracted from the vector

$$\widetilde{d_j} = \phi * \phi * \cdots * \phi * \psi * s,$$

where the number of convolutions with ϕ is $j-1$, and $\widetilde{d_j}$ is a vector containing all of the coefficients $d_{j,k}$ and MORE. The extracted coefficients to obtain $d_{j,k}$ are every 2^j elements of this vector, a process known as down sampling. Hence we can write the set of coefficients as multiplying this vector by a row sampling matrix P.

To this end, the fast wavelet transform is churned out by the following set of operations

$$\begin{aligned} s_0 &= s \\ d_j &= P\Psi s_{j-1} \\ s_j &= P\Phi s_{j-1} \end{aligned} \tag{16}$$

Here, d_j represents the vector of the full set of coefficients at level j. P is the row selector that selects every other entry of the input vector (hence it is changing size for each j), and Ψ and Φ are operators that perform the high pass and low pass filtering operations (hence convolution with ψ and ϕ).

Then we may write the full orthonormal transformation as

$$c = Ws = \begin{pmatrix} P\Psi \\ P\Psi P\Phi \\ P\Psi (P\Phi)^2 \\ \vdots \\ P\Psi (P\Phi)^{\ell-1} \\ (P\Phi)^{\ell} \end{pmatrix} s = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_\ell \\ s_\ell \end{pmatrix}$$

Note that I am abusing some notation here since P, Φ and Ψ will change size after each down sampling. I suppose the proper way to write it would be something like

$$W = \begin{pmatrix} P_N \Psi_N \\ P_{N/2} \Psi_{N/2} P_N \Phi_N \\ P_{N/4} \Psi_{N/4} P_{N/2} \Phi_{N/2} P_N \Phi_N \\ \vdots \end{pmatrix},$$

where the subscripts denote the dimension of the signal the operator is acting on.

5.3 Wavelet Reconstruction Algorithm

Since W is orthonormal, for the reconstruction we just need to evaluate

$$s = W^T c,$$

which is performed in a similar manner with up-sampling operations (P^T is a row *stretching* operation). It turns out though that we can do better than this. Notice for instance, the last two operations applied to each term will be

$$\Phi_N^T P_N^T \alpha_j,$$

for some α_j, and then all of these j terms will be added together. Hence, we should just add them together first and then apply this operation. Now generalize this idea, so that each of these upsampling and convolutions only happens once on a summed vector, and now you have a fast wavelet reconstruction algorithm.

5.4 Multidimensional Wavelets

The extension of wavelets from 1D to higher dimensions is straightforward (unless you want something more sophisticated, like shearlets). Observe, if you have two orthonormal bases in 1D, $\{u_j(x)\}_j$ and $\{v_j(x)\}_j$, then an orthonormal basis in 2D may be obtained by

$$\varphi_{jk}(x, y) = u_j(x)v_k(y),$$

over all combinations of j and k. The fast transformation algorithm is then easily modified by crossing the wavelet filters in two dimensions, and downsampling in both dimensions. It is also straightforward to mix the chosen wavelets bases in the different dimensions.

Appendix

proof of Proposition 1. (ii)⇒ (i). First note the following

$$\begin{aligned} \left\| \sum_n a_n \phi(x-n) \right\|_2^2 &= \left\| \sum_n a_n e^{-in\xi} \hat{\phi}(\xi) \right\|_2^2 \\ &= \int_{\mathbb{R}} \left| \sum_n a_n e^{-in\xi} \right|^2 \left| \hat{\phi}(\xi) \right|_2^2 \mathrm{d}\xi \\ &= \int_0^{2\pi} \left| \sum_n a_n e^{-in\xi} \right|^2 \sum_{k\in\mathbb{Z}} \left| \hat{\phi}(\xi + 2\pi k) \right|^2 \mathrm{d}\xi \end{aligned} \tag{17}$$

Using the condition from (ii) and then noting that $e^{-in\xi}$ is an orthogonal basis for $L_2[0,2\pi)$, leads to

$$\begin{aligned}\left\|\sum_n a_n\phi(x-n)\right\|_2^2 &\le A^2/2\pi\int_0^{2\pi}\left|\sum_n a_n e^{-in\xi}\right|^2 \mathrm{d}\xi \\ &= A^2/2\pi\sum_m\sum_n a_m a_n\int_0^{2\pi} e^{i(m-n)\xi}\mathrm{d}\xi \\ &= A^2/2\pi\sum_m\sum_n a_m a_n 2\pi\delta_{mn} \\ &= A^2\left(\sum_{n\in\mathbb{Z}} a_n^2\right)\end{aligned} \tag{18}$$

The lower bound inequality is obtained likewise.

Next (i)⇒(ii). Define

$$A_\alpha := \left\{\xi\in[0,2\pi)\mid\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi+2\pi k)|^2>\alpha\right\}.$$

Let

$$\mathbb{1}_{A_\alpha}(\xi)=\sum_{n\in\mathbb{Z}} a_n e^{-in\xi},\ \text{ where } a_n=\langle\mathbb{1}_{A_\alpha}(\xi),e^{-in\xi}/2\pi\rangle.$$

Then by (i)

$$\begin{aligned}A^2\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right) &\ge \left\|\sum_{n\in\mathbb{Z}} a_n\phi(x-n)\right\|_2^2 \\ &= \int_0^{2\pi}\left|\sum_{n\in\mathbb{Z}} a_n e^{-in\xi}\right|^2\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi+2\pi k)|^2\,\mathrm{d}\xi \\ &= \int_{A_\alpha}\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi+2\pi k)|^2\,\mathrm{d}\xi \\ &> \alpha|A_\alpha|.\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}\left(\int_0^{2\pi}\mathbb{1}_{A_\alpha}^2(\xi)\mathrm{d}\xi\right)^{1/2} &= |A_\alpha|^{1/2} \\ &= \left(\int_0^{2\pi}\left|\sum_n a_n e^{-in\xi}\right|^2\mathrm{d}\xi\right)^{1/2} \\ &= \sqrt{2\pi}\left(\sum_n |a_n|^2\right)^{1/2}\end{aligned} \tag{20}$$

Combining (19) and (20) obtains

$$\alpha|A_\alpha| < A^2\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right) = A^2 2\pi|A_\alpha|, \tag{21}$$

hence

$$\alpha < A^2/2\pi,$$

which completes the proof (for the other inequality just consider the reverse set of point less than α). □

proof of Proposition 3. The conditions of the statement imply

$$|u||x|e^{i(\theta_u-\theta_x)} + |v||y|e^{i(\theta_v-\theta_y)} = 0.$$

Taking the magnitude of this equation and rearranging yields [1]

$$\frac{|u|}{|v|} = \frac{|y|}{|x|}.$$

This implies there's a scalar $\alpha > 0$ so that

$$(|u|, |v|) = \alpha(|y|, |x|).$$

Now we search for θ_α so that

$$(u, v) = \alpha e^{i\theta_\alpha}(\overline{y}, -\overline{x})$$

Then to get the phases of the first part of the equation to match, $u = \alpha e^{i\theta_\alpha}\overline{y}$, we need $\theta_\alpha = \theta_u + \theta_y$. Similarly, for the second part of the equation, $\theta_\alpha = \theta_v + \theta_x + \pi$. From the first line of the proof, observe that

$$\theta_\alpha = \theta_u + \theta_y = \theta_v + \theta_x + \pi.$$

□

proof of Proposition 4. Note that $f \in W_0 \iff f \in V_1, f \perp V_0$. Then $f \perp V_0$ if and only if for every n

$$\begin{aligned} 0 &= \langle f, \phi(x-n)\rangle \\ &= \langle \hat{f}, e^{-i\xi n}\hat{\phi}(\xi)\rangle \\ &= \int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi n}\overline{\hat{\phi}(\xi)}\,\mathrm{d}\xi \\ &= \int_0^{2\pi} e^{i\xi n}\sum_{k\in\mathbb{Z}}\hat{f}(\xi+2\pi k)\overline{\hat{\phi}(\xi+2\pi k)}\,\mathrm{d}\xi \\ &= \int_0^{2\pi} e^{i\xi n}F(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{22}$$

[1] Assuming no values of zero, in which case the proposition is trivial.

Notice that $F(\xi)$ is 2π periodic, but since the last line is 0, $F(\xi) = 0$ almost everywhere. Substituting in the scaling equations for ϕ and for f in (11) leads to

$$\begin{aligned}
0 &= \sum_{k\in\mathbb{Z}} m_f(\xi/2+\pi k)\overline{m_\phi(\xi/2+\pi k)}\,|\hat{\phi}(\xi/2+\pi k)|^2 \\
&= \sum_{k\in\mathbb{Z}} m_f(\xi/2+2\pi k)\overline{m_\phi(\xi/2+2\pi k)}\,|\hat{\phi}(\xi/2+2\pi k)|^2 \\
&+ \sum_{k\in\mathbb{Z}} m_f(\xi/2+2\pi k+\pi)\overline{m_\phi(\xi/2+2\pi k+\pi)}\,|\hat{\phi}(\xi/2+2\pi k+\pi)|^2 \\
&= m_f(\xi/2)\overline{m_\phi(\xi/2)}\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi/2+2\pi k)|^2 \\
&+ m_f(\xi/2+\pi)\overline{m_\phi(\xi/2+\pi)}\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi/2+2\pi k+\pi)|^2 \\
&= (2\pi)^{-1}\left[m_f(\xi/2)\overline{m_\phi(\xi/2)} + m_f(\xi/2+\pi)\overline{m_\phi(\xi/2+\pi)}\right]
\end{aligned} \tag{23}$$

Hence letting $\eta = \xi/2$

$$\begin{aligned}
0 &= \left[m_f(\eta)\overline{m_\phi(\eta)} + m_f(\eta+\pi)\overline{m_\phi(\eta+\pi)}\right] \\
&= \langle (m_f(\eta), m_f(\eta+\pi)), (m_\phi(\eta), m_\phi(\eta+\pi))\rangle \\
&\Rightarrow (m_f(\eta), m_f(\eta+\pi)) \perp (m_\phi(\eta), m_\phi(\eta+\pi))
\end{aligned} \tag{24}$$

This implies (see Proposition 3)

$$(m_f(\eta), m_f(\eta+\pi)) = \alpha(\eta)(\overline{m_\phi(\eta+\pi)}, -\overline{m_\phi(\eta)}). \tag{25}$$

Hence,

$$(m_f(\eta+\pi), m_f(\eta)) = \alpha(\eta+\pi)(\overline{m_\phi(\eta)}, -\overline{m_\phi(\eta+\pi)}). \tag{26}$$

By (25) and (26) we see that

$$\begin{aligned}
m_f(\eta) &= \alpha(\eta)\overline{m_\phi(\eta+\pi)} \\
m_f(\eta) &= -\alpha(\eta+\pi)\overline{m_\phi(\eta+\pi)} \\
\to \alpha(\eta) &= -\alpha(\eta+\pi)
\end{aligned} \tag{27}$$

Then

$$h(\eta) := e^{-i\eta}\alpha(\eta)$$

is π-periodic. Hence

$$\begin{aligned}
\hat{f}(\xi) &= m_f(\xi/2)\hat{\phi}(\xi/2) \\
&= e^{i\xi/2}\underbrace{h(\xi/2)}_{V(\xi)}\overline{m_\phi(\xi/2+\pi)}\hat{\phi}(\xi/2) \\
&= e^{i\xi/2}V(\xi)\overline{m_\phi(\xi/2+\pi)}\hat{\phi}(\xi/2)
\end{aligned} \tag{28}$$

This completes the first statement of the proposition.

Recall, $\hat{f}(\xi) = m_f(\xi/2)\hat{\phi}(\xi/2)$ and

$$\frac{1}{\sqrt{2}}\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |m_f(\xi)|^2 \mathrm{d}\xi\right)^{1/2}.$$

By what we have already proven

$$m_f(\xi/2) = e^{i\xi/2}V(\xi)\overline{m_\phi(\xi/2+\pi)},$$

hence

$$\|f\|_2 = \sqrt{2}\left(\frac{1}{2\pi}\int_0^{2\pi} |V(2\xi)|^2 |m_\phi(\xi+\pi)|^2 \mathrm{d}\xi\right)^{1/2}.$$

Since $V(2\xi)$ is π-periodic and m_ϕ is 2π periodic we have

$$\|f\|_2 = \sqrt{2}\left(\frac{1}{2\pi}\int_0^{\pi} |V(2\xi)|^2 (|m_\phi(\xi+\pi)|^2 + |m_\phi(\xi)|^2) \mathrm{d}\xi\right)^{1/2}$$

Using Lemma 1 completes the proof. □