The Radon Transform

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December 2012

Introduction

The Radon transform is the integral transform, which in $\mathbb{R}^d$ consists of the integral of a function f over all $d-1$ dimensional hyperplanes. An important problem in image processing is the reconstruction problem. That is, can one recover the function f given the Radon transform, $\mathfrak{R}f$? Being able to provide a fast and accurate solution to this problem is vital in X-ray computed tomography (CT), ultrasound CT, astronomy, and electron microscopy, to name a few. For example, in an X-ray or MRI scan, the function f we are interested in is the image of the object being scanned. Whenever the scanning process occurs, only the values of the Radon transform of f are found. Thus we need to be able to accurately recover f from only some knowledge of it's Radon transform. In this paper we will give the solution to this problem, along with some applicable tools for reconstruction.

Definition of the Radon Transform

In $\mathbb{R}^2$ we may write any line L uniquely as the set of ordered pairs (x, y) satisfying $t = x\cos\omega + y\sin\omega$, for some fixed $t \in [0, \infty)$ and $\omega \in [0, 2\pi)$. Here and throughout the paper we let ξ denote a unit vector in $\mathbb{R}^2$, and in particular $\xi := (\cos\omega, \sin\omega)$. Then we write the line L as the set of (x, y) satisfying $\mathbf{x} \cdot \xi = t$, where $\mathbf{x}$ is defined to be (x, y). Then we have the Radon transform

$$\mathfrak{R}f := \check{f}(t, \xi) := \int_{\mathbf{x}\cdot\xi=t} f(x, y)ds,$$

i.e. the integral of f over lines in $\mathbb{R}^2$. Now, one can verify that we may write the line L parametrically as a function of s as

$$\begin{aligned} x &= t\cos\omega - s\sin\omega \\ y &= t\sin\omega + s\cos\omega. \end{aligned}$$

Therefore, we may write the Radon transform as

$$\begin{aligned} \check{f}(t, \xi) &= \int_{-\infty}^{\infty} f(t\cos\omega - s\sin\omega, t\sin\omega + s\cos\omega)ds \\ &= \int_{-\infty}^{\infty} f(t\xi + s\xi^{\perp})ds, \end{aligned}$$

where $\xi^{\perp} = (-\sin\omega, \cos\omega)$. In many practical applications we do not need to integrate over the entire space, since our function f will have finite support.
Finally, we have one more equivalent form given by

$$\check{f}(t, \xi) = \iint_{\mathbb{R}^2} f(x, y)\delta(t - \xi \cdot \mathbf{x})dxdy,$$

where δ is the Dirac delta function. This form can be useful when showing how the Radon transform behaves under translations and dilations of f.

Example 1. Let $f(x, y) = e^{-x^2-y^2}$. Then,

$$\check{f}(t, \xi) = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\delta(t - \xi \cdot \mathbf{x})dxdy.$$

Now making the orthogonal linear transformation

$$\begin{aligned} u &= \xi \cdot \mathbf{x} \\ v &= \xi^{\perp} \cdot \mathbf{x} \end{aligned}$$

we have

$$\begin{aligned} \check{f}(t, \xi) &= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{-u^2-v^2}\delta(t - u)dudv \\ &= e^{-t^2}\int_{-\infty}^{\infty} e^{-v^2}dv \\ &= \sqrt{\pi}e^{-t^2}. \end{aligned}$$

Using a similar transformation in higher dimensions for the function $f(x_1, \ldots, x_d) = exp(-x_1^2 + \cdots + -x_n^2)$ yields

$$\check{f}(t, \xi) = (\sqrt{\pi})^{n-1}e^{-t^2}$$

Relation to the Fourier Transform

The Fourier transform of $f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^d$, is defined as

$$\mathfrak{F}f := \hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} f(\mathbf{x})e^{-i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.$$

We do not go into detail here with the appropriate space of functions on which $\mathfrak{F}$ is defined. We simply assume that f is nice enough (for example, f may be a Shcwartz function). It well known that the inverse Fourier transform exists and is given by

$$\mathfrak{F}^{-1}\hat{f} := f(\mathbf{x}) = \int_{\mathbb{R}^d} \hat{f}(\mathbf{k})e^{i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}.$$

To connect the Fourier transform with the Radon transform, we first observe that we may rewrite the Fourier transform as

$$\hat{f}(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} f(\mathbf{x})e^{-i2\pi t}\delta(t - \mathbf{k}\cdot\mathbf{x})d\mathbf{x}dt,$$

and for any $s \neq 0$ we have $\delta(sx) = \frac{\delta(x)}{|s|}$, which the reader may verify. Then

$$\begin{aligned} \hat{f}(s\xi) &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} f(\mathbf{x})e^{-i2\pi t}\delta(t - s\xi\cdot\mathbf{x})d\mathbf{x}dt \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} f(\mathbf{x})e^{-i2\pi sp}\delta(sp - s\xi\cdot\mathbf{x})d\mathbf{x}dp|s| \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} f(\mathbf{x})e^{-i2\pi sp}\delta(p - \xi\cdot\mathbf{x})d\mathbf{x}dp \\ &= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^d} f(\mathbf{x})\delta(p - \xi\cdot\mathbf{x})d\mathbf{x}\right)e^{-i2\pi sp}dp. \end{aligned}$$

Notice that the inner integral is just the Radon transform of f. Thus we have established the following relationship between the transforms:

$$\hat{f}(s\xi) = \int_{-\infty}^{\infty} \check{f}(t,\xi) e^{-i2\pi st} dt$$

Since the Fourier transform is invertible, this identity at the very least shows the existence of f, but does not give a simple formula for recovering f. Explicit inversion formulas (depending on whether the dimension is even or odd) do exist, but we will not show them here. Instead the remainder of the paper will focus on building some practical tools that may be used for the reconstruction. We digress here to mention some classical results in approximation theory. These results will be useful for what we need.

Some Classical Approximation Theory Results

In this section and the next we write D to represent the unit disk in $\mathbb{R}^2$.

Definition 2. We say the sequence of functions $\{K_n\}_{n=1}^{\infty}$ is a summability kernel if

$$\int_{\mathbb{R}} K_n(t)dt = 1 \quad \forall n$$

and for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_{|t|>\varepsilon} K_n(t)dt = 0.$$

Theorem 3. *Let $\{K_n\}_{n=1}^{\infty}$ be a summability kernel and let $f \in L_p(\mathbb{R})$. Then*

$$\|K_n * f - f\|_p \longrightarrow 0$$

and

$$K_n * f(x) \longrightarrow f(x) \quad a.e.,$$

as $n \to \infty$.

Proof. See lecture notes on 9/28/2012. ☐

Lemma 4. *The sequence of functions*

$$P_n(x,y) := \begin{cases} \frac{n+1}{\pi}(1-x^2-y^2)^n, & (x,y)\in\mathcal{D} \\ 0, & otherwise \end{cases}$$

$n = 1, 2, \ldots,$ is a summability kernel in $\mathbb{R}^2$.

Proof. First we have,

$$\begin{aligned}
\frac{\pi}{n+1}\iint_{\mathbb{R}^2} P_n(\mathbf{x})dxdy = \frac{\pi}{n+1}\iint_{\mathcal{D}} P_n(\mathbf{x})dxdy &= \iint_{\mathcal{D}} (1-x^2-y^2)^n dxdy \\
&= \iint_{\mathcal{D}} (1-r^2)^n r dr d\theta = \int_0^{2\pi}\int_0^1 \sum_{k=0}^{n} (-r^2)^k \binom{n}{k} r dr d\theta \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^{2\pi}\int_0^1 r^{2k+1} dr d\theta \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2\pi}{2k+2} = \pi \sum_{k=0}^{n} (-1)^k \frac{n!}{(k+1)!(n-k)!} \\
&= -\frac{\pi}{n+1}\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} = -\frac{\pi}{n+1}\left(\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} - 1\right) \\
&= -\frac{\pi}{n+1}\left((1-1)^{n+1} - 1\right) = \frac{\pi}{n+1}
\end{aligned}$$

Thus we have shown

$$\iint_{\mathbb{R}^2} P_n(\mathbf{x})dxdy = 1, \quad \text{for all } n. \tag{1}$$

Secondly, fix some $0 < \varepsilon < 1$. Then similar to the computations above we have,

$$\begin{aligned}
\frac{\pi}{n+1}\iint_{|\mathbf{x}|>\varepsilon} P_n(\mathbf{x})dxdy &= \iint_{\varepsilon<|\mathbf{x}|<1}(1-x^2-y^2)^n dxdy \\
&= \int_0^{2\pi}\int_\varepsilon^1 (1-r^2)^n r dr d\theta \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2\pi}{2k+2}(1-\varepsilon^{2k+2}) \\
&= \frac{\pi}{n+1}\left(\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k} + \sum_{k=1}^{n+1}(-\varepsilon^2)^k\binom{n+1}{k}\right) \\
&= \frac{\pi}{n+1}\left(-\sum_{k=0}^{n+1}(-1)^{k}\binom{n+1}{k} + \sum_{k=0}^{n+1}(-\varepsilon^2)^k\binom{n+1}{k} + 1 - 1\right) \\
&= \frac{\pi}{n+1}\left(-(1-1)^{n+1} + (1-\varepsilon^2)^{n+1}\right) \\
&= \frac{\pi}{n+1}(1-\varepsilon^2)^{n+1}.
\end{aligned}$$

This computation shows

$$\iint_{|\mathbf{x}|>\varepsilon} P_n(\mathbf{x})dxdy = (1-\varepsilon^2)^{n+1} \longrightarrow 0 \quad \text{as } n\to\infty. \tag{2}$$

Thus (1) and (2) together complete the proof. □

Theorem 5 (Weierstrauss approximation theorem)**.** *The set of all 2 dimensional polynomials over the disk is dense in the space $C(\mathcal{D})$. That is, for all $\varepsilon > 0$ and for any $f \in C(\mathcal{D})$ there exists n such that*

$$\inf_{P\in\mathbb{P}_n(\mathcal{D})}\|f-P\|_{C(\mathcal{D})} < \varepsilon.$$

Proof. We may extend f over all of $\mathbb{R}^2$ so that we have $f \in C_(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$. Now, let $\{P_n\}_{n=1}^\infty$ be the sequence of polynomials defined in lemma 4, and consider the sequence of functions $\{f_n\}_{n=1}^\infty$, where

$$f_n(\mathbf{x}) := f * P_n(\mathbf{x}) = \iint_{\mathcal{D}} f(\mathbf{y})P_n(\mathbf{x}-\mathbf{y})d\mathbf{y}.$$

Then since $\{P_n\}_{n=1}^\infty$ is a summability kernel, it follows from theorem 3 that $f_n(\mathbf{x})$ converges to $f(\mathbf{x})$ almost everywhere as n tends to ∞. Therefore, on the compact disk $\mathcal{D}$, f_n converges to f uniformly. Hence, there exists n such that

$$\|f-f_n\|_{C(\mathcal{D})} < \varepsilon \tag{3}$$

for any given $\varepsilon > 0$. Lastly, we note that by definition, f_n must be a polynomial of degree $2n$ over $\mathcal{D}$, and the theorem then follow from (3). □

Theorem 6. *The Chebyshev polynomial $T_n(x) := \cos(n \arccos x)$ is equivalent to a polynomial of degree n.*

Proof. The proof is by induction. Certainly we have $T_0(x) = 1$ and $T_1(x) = x$.
Claim: $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.
Well, using some simple trigonometric identities we have

$$\begin{aligned} T_n(x) + T_{n-2}(x) &= \cos(n \arccos x) + \cos((n-2) \arccos x) \\ &= 2 \cos(\arccos x) \cos((n-1) \arccos x) \\ &= 2xT_{n-1}(x). \end{aligned}$$

This identity completes the proof. □

Approximation using ridge polynomials

In this section, we will find properties of ridge polynomials and show how they may be used to reconstruct a function from its (semi-discrete) Radon transform. If $f, g \in L_2(\mathcal{D})$, we define the inner product

$$\langle f, g \rangle := \frac{1}{\pi} \iint_{\mathcal{D}} f(\mathbf{x}) g(\bar{\mathbf{x}}) d\mathbf{x},$$

which also induces the norm in $L_2(\mathcal{D})$. We will be using the Chebyshev polynomials of second kind for approximation in $L_2(\mathcal{D})$. These functions are given by

$$U_m(t) := \frac{\sin(m \arccos t)}{\sin(\arccos t)}, \quad m = 1, 2, \dots, .$$

Certainly these are only univariate functions. However, we let $\Omega_m := \{k\pi/m : k = 1, 2, \dots, m\}$ and instead use the functions

$$U_m^\omega(\mathbf{x}) := U_m(\mathbf{x} \cdot \xi), \quad \mathbf{x} \in \mathcal{D} \quad \omega \in \Omega_m,$$

for approximation, where $\xi := (\cos \omega, \sin \omega)$.
By differentiation of the Chebyshev polynomial $T_m(t)$ defined in theorem 6, one finds that $T_m^{'}(t) = U_m(t)$. Therefore, $U_m(t)$ is equivalent to a polynomial of degree $m-1$, thus $U_m^\omega \in \mathbb{P}_{m-1}(\mathcal{D})$, where $\mathbb{P}_{m-1}(\mathcal{D})$ denotes the set of polynomials of degree $m-1$ over the disk. This fact will be very important in showing that these functions may be used for approximation over the disk.

Lemma 7. *The set of functions*

$$\{U_m^\omega : \omega \in \Omega_m, \quad m = 1, 2, \dots, \}$$

are orthonormal with respect to the inner product on $L_2(\mathcal{D})$, i.e.

$$\langle U_m^\alpha, U_n^\beta \rangle = \delta_{m,n} \delta_{\alpha,\beta}.$$

Proof. This proof will involve showing two facts. We will show first that

$$\int_{\mathcal{D}} U_m^\alpha U_m^\beta d\mathbf{x} = \delta_{\alpha,\beta} \quad \text{for } \alpha, \beta \in \Omega_m, \tag{4}$$

and secondly that

$$\iint_{\mathcal{D}} x^k y^j U_m^\omega dx dy = 0 \quad \text{for } k + j < m - 1, \text{ and } \omega \in \Omega_m. \tag{5}$$

We see that (5) says U_m^ω is orthogonal to all polynomials of degree less than $m-1$ over $\mathcal{D}$. Recalling that U_m^ω is a polynomial of degree $m-1$, (5) holding true tells us that if $n < m$ that $\langle U_n^\alpha, U_m^\beta \rangle = 0$, regardless of α and β. To this end, showing (4) and (5) will complete the proof.

To begin note that any integral over $\mathcal{D}$ is rotational invariant, so without loss of generality we will assume $\beta = 0$. Then we have

$$\begin{aligned}\int_{\mathcal{D}} U_m^{\alpha} U_m^{\beta} dydx &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} U_m(x\cos\alpha + y\sin\alpha) U_m(x) dydx \\ &= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} U_m(x\cos\alpha + y\sin\alpha) dy \right) U_m(x) dx.\end{aligned}$$

We will compute the inner integral with respect to y first. We make the substitution $x = \cos t$ for x, then $u = \cos t \cos\alpha + y \sin\alpha$ for y, and finally $u = \cos v$ for u:

$$\begin{aligned}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} U_m(x\cos\alpha + y\sin\alpha) dy &= \int_{-\sin t}^{\sin t} U_m(\cos t\cos\alpha + y\sin\alpha) dy \\ &= \frac{1}{\sin\alpha} \int_{\cos(t+\alpha)}^{\cos(t-\alpha)} U_m(u) du \\ &= \frac{1}{\sin\alpha} \int_{t-\alpha}^{t+\alpha} U_m(\cos v) sin(v) dv \\ &= \frac{1}{\sin\alpha} \int_{t-\alpha}^{t+\alpha} \sin nv dv \\ &= \frac{2\sin(mt)\sin(m\alpha)}{m\sin\alpha}.\end{aligned}$$

The last line follows from simple computations and trigonometric identities. Now, returning back to the original integral, and keeping the substitution $x = \cos t$ we have

$$\begin{aligned}\iint_{\mathcal{D}} U_m^{\alpha} U_m^{0} dxdy &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} U_m(x\cos\alpha + y\sin\alpha) U_m(x) dydx \\ &= \int_{0}^{\pi} \frac{2\sin(mt)\sin(m\alpha)}{m\sin\alpha} U_m(\cos t) \sin t dt \\ &= \frac{2\sin(m\alpha)}{m\sin\alpha)} \int_0^{\pi} \sin^2 mt dt \\ &= \pi \frac{\sin(m\alpha)}{m\sin\alpha},\end{aligned}$$

where again the last line follow by some simple computations. Now notice that the above is 0 for $\alpha = \frac{k\pi}{n}, \quad k = 1, 2, \ldots, n-1$ and π for $\alpha = 0$, which was precisely what we wanted.

Now to show (5) we will again use the fact that the integral of the disk is rotational invariant. So, without loss of generality, let $\omega = 0$ and $k + j < m - 1$. Then we have

$$\begin{aligned}\iint_{\mathcal{D}} x^k y^j U_m^{\omega} dydx &= \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^k y^j U_m(x) dydx \\ &= \frac{1}{j+1} \int_{-1}^{1} x^k U_m(x) \left((1-x^2)^{(j+1)/2} - (-1)^{j+1} (1-x^2)^{(j+1)/2} \right) dx.\end{aligned}$$

From here we have 2 cases. If j is odd, then

$$\iint_{\mathcal{D}} x^k y^j U_m^{\omega} dydx = \frac{1}{j+1} \int_{-1}^{1} x^k U_m(x) \left((1-x^2)^{(j+1)/2} - (1-x^2)^{(j+1)/2} \right) dx = 0.$$

If j is even, we again will make the useful substitution $x = \cos t$, and we have

$$\begin{aligned}\iint_{\mathcal{D}} x^k y^j U_m^\omega dydx &= \frac{2}{j+1}\int_{-1}^{1} x^k (1-x^2)^{j/2}\sqrt{1-x^2}U_m(x)dx \\ &= \frac{2}{j+1}\int_{-1}^{1} x^k \left(\sum_{p=0}^{j/2}(-1)^p x^{2p}\binom{j/2}{p}\right)\sqrt{1-x^2}U_m(x)dx \\ &= \frac{2}{j+1}\sum_{p=0}^{j/2}(-1)^p\binom{j/2}{p}\int_{-1}^{1} x^{2p+k}\sqrt{1-x^2}\frac{\sin(m\arccos x)}{\sin(\arccos x)}dx \\ &= \frac{2}{j+1}\sum_{p=0}^{j/2}(-1)^p\binom{j/2}{p}\int_{0}^{\pi}(\cos t)^{2p+k}\cos t\frac{\sin mt}{\sin t}\sin tdt \\ &= \frac{2}{j+1}\sum_{p=0}^{j/2}(-1)^p\binom{j/2}{p}\int_{0}^{\pi}(\cos t)^{2p+k+1}\sin mtdt.\end{aligned}$$

Now, note that $(\cos t)^{2p+k+1}$ is a trigonometric polynomial of degree $2p+k+1 \leq j+k+1 < m$. And it can easily be shown that if $T_n(t)$ is a polynomial of degree $n < m$, then $\int_0^\pi T_n(t)\sin mtdt = 0$. Thus the integral is 0 for j even as well, and the proof is complete. □

Since $U_m^\omega \in \mathbb{P}_{m-1}(\mathcal{D})$, we see that $span\{A_n\} \subset \mathbb{P}_{n-1}(\mathbb{R}^2)$, where

$$A_n := \{U_m^\omega : \omega \in \Omega_m, \quad m = 1, 2, \ldots, n\},$$

Moreover, by lemma 7 we see that the functions in A_n for an orthonormal system for $\mathbb{P}_{n-1}(\mathcal{D})$. Lastly, one may easily find that $dim(\mathbb{P}_{n-1}(D)) = \#A_n$. Therefore, we have that the set of functions in A_n form an orthonormal basis for $\mathbb{P}_{n-1}(\mathbb{R}^2)$. Since we have already shown the density of the polynomials over the disk, we have proven the following theorem.

Theorem 8. *Any function $f \in L_2(\mathcal{D})$ can be represented as*

$$f(\mathbf{x}) = \sum_{m=1}^{\infty}\sum_{\omega\in\Omega_m}\langle f, U_m^\omega\rangle U_m^\omega(\mathbf{x}). \tag{6}$$

Proof. Since $\{U_m^\omega : \omega \in \Omega_m, \quad m = 1, 2, \ldots,\}$ forms an orthonormal basis for $\mathbb{P}_{n-1}(\mathcal{D})$, the result follows directly from the Weierstrass approximation theorem. □

Reconstruction using Ridge functions

We will show how the representation (6) of a function f on the unit disk can be used to reconstruct a function from its Radon transform. This involves just a few simple computations. Let $f \in L_2(\mathcal{D})$, and to simplify computations, extend f and U_m^ω to be 0 for $\mathbf{x} \notin \mathcal{D}$. Again let $\xi = (\cos\omega, \sin\omega)$ and assume $\omega \in \Omega_m\backslash\{\pi\}$

then

$$\begin{aligned}
\pi\langle f, U_m^\omega\rangle &= \iint_{\mathcal{D}} f(\mathbf{x}) U_m(\mathbf{x}\cdot\xi) dy dx \\
&= \iint_{\mathbb{R}^2} f(\mathbf{x}) U_m(\mathbf{x}\cdot\xi) dy dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(p\sin\omega, \frac{t - p\sin\omega\cos\omega}{\sin\omega}\right) U_m(t) dp dt \\
&= \int_{\infty}^{\infty}\left(\int_{\mathbf{x}\cdot\xi=t} f(x,y) ds\right) U_m(t) dt \\
&= \int_{-\infty}^{\infty} \check{f}(t,\xi) U_m(t) dt \\
&= \int_{-1}^{1} \check{f}(t,\xi) U_m(t) dt,
\end{aligned}$$

where we applied the substitution $t = \mathbf{x}\cdot\xi$ for y, and $x = p\sin\omega$ for x. In the case $\omega = \pi$, performing similar computations, we obtain the same result. Substituting these expressions into (6) gives

$$f(\mathbf{x}) = \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{\omega\in\Omega_m}\left(\int_{-1}^{1}\check{f}(t,\xi) U_m(t) dt\right) U_m^\omega(\mathbf{x}).$$

Thus this expression gives us a way to recover a function $f \in L_2(\mathcal{D})$ given the Radon transform of f.