A Brief Discussion: Laplacian Image Sharpening versus Deconvolution - one in the same

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Abstract

In this short document, I begin by briefly introduction image sharpening and comparing this with deconvolution with Wiener filtering. Then I show how the two could be made equivalent, i.e. given a particular sharpening method, there exists a PSF such that deconvolution with that PSF via Wiener filtering is equivalent to the sharpening. Even more striking, I show that Laplacian image sharping (sharpening with the Laplace operator) is approximately equivalent to Wiener filtering with a PSF taken from a Laplacian distribution. It is simply a coincidence that the name "Laplace" shows up in these equivalent methods. I am unaware if any of this is original work, but I was unable to locate any of it.

1 Image Sharpening Vs. Deconvolution

A common image sharpening and edge enhancement technique is one which adds in a high pass filter of the image to the original image. This is often call unsharp masking for one reason or another, but I refer to it as Laplacian sharpening. To describe it, let's call the original image say u and let Δ denote the Laplace operator. Since the Laplace operator is a differential operator, its application can be represented by convolution with a kernel, we will call g. Then the sharpened image, v, takes the following forms

$$\begin{aligned} v(\alpha) &= u + \alpha \Delta u \\ &= u + \alpha g * u \\ &= (\delta + \alpha g) * u \\ &= \mathcal{F}^{-1}\{(1 + \alpha \hat{g}) \cdot \hat{u}\}, \end{aligned} \tag{1}$$

where $\mathcal{F}$ is the 2D discrete Fourier transform, and the "hats" denote the Fourier transforms of the images. The parameter $\alpha > 0$ determines the amount of sharpening, and typically takes on values roughly in the interval $[0, 3]$ in my experience.

An alternative to image sharpening is deconvolution, which are both forms of high pass filters. Deconvolution is more well founded for restoring images known to contain blur, while sharpening is typically used to visually enhance ordinary image that may or may not be corrupted. For deconvolution problems, the image data is assumed to take the form

$$u = v * h + \epsilon,$$

where h is some point spread function (PSF) that has blurred or "unsharpened" v. Then v may be recovered or estimated by deconvolution. For example, a simple Wiener deconvolution (in Fourier domain representation) takes the form

$$\hat{v} = \hat{u} \cdot \frac{\overline{\hat{h}}}{|\hat{h}|^2 + \lambda}, \tag{2}$$

which can be formally derived from various models in various ways.

Below, I argue that Wiener deconvolution and Laplacian sharpening are essentially equivalent. Let's take an ordinary Wiener deconvolution of an image u given a point spread function h. Comparing the Wiener filter in (2) with that in the last line of (1), the Wiener filter and sharpening filter are equivalent if

$$(1+\alpha\hat{g}) = \frac{\overline{\hat{h}}}{|\hat{h}|^2+\lambda},$$

or equivalently

$$(1+\alpha\hat{g})|\hat{h}|^2 - \overline{\hat{h}} + \lambda(1+\alpha\hat{g}) = 0. \tag{3}$$

Letting

$$\begin{aligned} a &= (1+\alpha\hat{g}) \\ b &= -1 \\ c &= \lambda(1+\alpha\hat{g}), \end{aligned} \tag{4}$$

then (3) is re-written once more as

$$a|\hat{h}|^2 + b\overline{\hat{h}} + c = 0, \tag{5}$$

which looks a lot like an ordinary quadratic equation. Hence, given a chosen α and λ, we can just use the quadratic formula to find h. If we define $z = 1 + alpha\hat{g}$, then this leads to h as

$$\hat{h} = \frac{1}{2z} \pm \frac{\sqrt{1-4z^2\lambda}}{2z}.$$

However, there are a few constraints to consider. We need $\hat{h}$ to be real valued. This is because the Laplacian filter is symmetric, hence for the equivalence to hold, h should also be symmetric. If h is symmetric then is Fourier transform is real. Looking back at the solution for $\hat{h}$ and using the quadratic formula, this imposes the constraint

$$\lambda(1+\alpha\hat{g})^2 \leq 1/4.$$

This is interesting because I am accustomed to having no regularization when Laplacian sharpening, and the constraint listed above says we can find a corresponding Wiener filter so long as α and λ are the appropriate sizes. However, after more thought, a Laplacian sharpening does implicitly contain *some* regularization when compared with a standard deconvolution with no regularization, because the Laplacian sharpening doesn't *blow up*, figuratively speaking, whereas dividing by the PSF directly to deconvolve typically would.

1.1 Laplace Sharpening leads to Wiener Filtering with a Laplace Distribution

Let $z = 1 + \alpha\hat{g}$. Then looking at (3) and (5) we see that

$$\hat{h} = \frac{1}{2z} \pm \frac{\sqrt{1-4z^2\lambda}}{2z} = \frac{1}{2z} \pm \sqrt{\frac{1}{4z^2} - \lambda}.$$

If λ and z are small, then the first solution is approximately

$$\hat{h} \approx \frac{1}{z}.$$

It can be shown that for the 1D case (and more or less the same in 2D) that we have the Fourier transform pair

$$g = [-1, 2, -1] \longleftrightarrow \hat{g}_\xi = 4\sin^2(\pi\xi/N).$$

To show this, we just need to make sure g is centered properly so it is symmetric and $\hat{g}$ is real, and so forth. In any case, this would make

$$\hat{h}_\xi \approx \frac{1}{1+4\alpha\sin^2(\pi\xi/N)}.$$

I believe this may be the Fourier transform of a discrete Laplacian distribution. In the continuous case, the Fourier transform of a discrete Laplacian distribution (ignoring the constants and variances) is roughly

$$\hat{f}(\xi) = \frac{1}{1+\xi^2},$$

so it's close at least.

Here goes the formal proof... let $z \in \mathbb{R}^N$ be given by

$$z(x) = e^{-\beta|x|}, \quad \text{for} \quad x = -N/2, -N/2+1, \ldots, N/2-1.$$

Then

$$\begin{aligned}\hat{z}_k &= \sum_{x=-N/2}^{N/2-1} e^{-\beta|x|} e^{-i2\pi kx/N} \\ &= -1 + e^{-\beta N/2} e^{-i\pi k} + \sum_{x=0}^{N/2-1} e^{-\beta x} \left(e^{-i2\pi kx/N} + e^{i2\pi kx/N}\right)\end{aligned} \tag{6}$$

Here we have peeled the largest magnitude term, $x = -N/2$, out of the sum, and doubled the $x = 0$ term, which is gives rise to the extra -1. For N even only modestly large, we can set the sum to go to infinity, which only introduces error of $O(e^{-\beta N/2})$. This leads to

$$\begin{aligned}\hat{z}_k &\approx -1 + \sum_{x=0}^{\infty} e^{-\beta x} \left(e^{-i2\pi kx/N} + e^{i2\pi kx/N}\right) \\ &= -1 + \frac{1}{1 - e^{-(\beta + i2\pi k/N)}} + \frac{1}{1 - e^{-(\beta - i2\pi k/N)}}\end{aligned} \tag{7}$$

To simplify this, we first work out the common denominator and simplify:

$$\begin{aligned}&(1 - e^{-(\beta + i2\pi k/N)})(1 - e^{-(\beta - i2\pi k/N)}) \\ &= 1 + e^{-2\beta} - e^{-\beta}(e^{-i2\pi k/N} + e^{i2\pi k/N}) \\ &= 1 + e^{-2\beta} - e^{-\beta} \cdot 2 \cdot \cos(2\pi k/N) \\ &= 1 + e^{-2\beta} - 2e^{-\beta} + 2e^{-\beta} - e^{-\beta} \cdot 2 \cdot \cos(2\pi k/N) \\ &= (1 - e^{-\beta})^2 + 4e^{-\beta} \sin^2(\pi k/N)\end{aligned} \tag{8}$$

In a similar fashion, we sort out the numerator:

$$\begin{aligned}&(1 - e^{-(\beta + i2\pi k/N)}) + (1 - e^{-(\beta - i2\pi k/N)}) \\ &= 2 - e^{-\beta}(e^{-i2\pi k/N} + e^{i2\pi k/N}) \\ &= 2 - 2e^{-\beta} \cos(2\pi k/N) \\ &= 2 - 2e^{-\beta} + 2e^{-\beta} - 2e^{-\beta} \cos(2\pi k/N) \\ &= 2(1 - e^{-\beta}) + 4e^{-\beta} \sin^2(\pi k/N)\end{aligned} \tag{9}$$

Putting it all together we have

$$\begin{aligned}\hat{z}_k &\approx -1 + \frac{2(1 - e^{-\beta}) + 4e^{-\beta} \sin^2(\pi k/N)}{(1 - e^{-1})^2 + 4e^{-\beta} \sin^2(2\pi k/N)} \\ &= \frac{2(1 - e^{-\beta}) - (1 - e^{-\beta})^2}{(1 - e^{-\beta})^2 + 4e^{-\beta} \sin^2(\pi k/N)} \\ &= \frac{1 - e^{-2\beta}}{(1 - e^{-\beta})^2 + 4e^{-\beta} \sin^2(\pi k/N)} \\ &= C \frac{1}{1 + 4\alpha \sin^2(\pi k/N)},\end{aligned} \tag{10}$$

where

$$C = \frac{1 - e^{-2\beta}}{(1 - e^{-\beta})^2}, \quad \alpha = \frac{e^{-\beta}}{(1 - e^{-\beta})^2}.$$

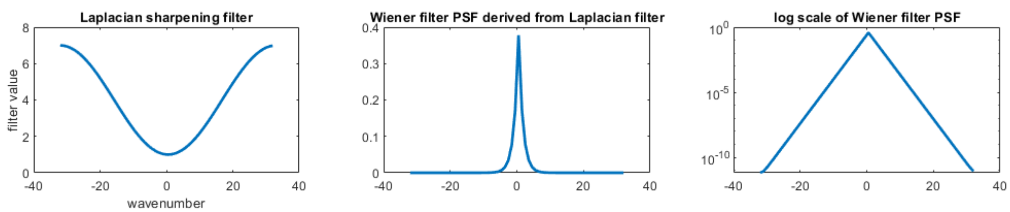

Figure 1: Example in 1D. Wiener filter PSF (middle and right) that is derived from a Laplacian sharpening filter (left). Interestingly, the log scale plot of the PSF shows that it is perhaps a Laplacian distribution.