Whitney's Theorem

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Theorem 1 (Whitney, 1957)**.** *Let $0 < p \leq \infty$, $f \in L_p[a,b]$, and $k \geq 1$. Then there exists a polynomial $Q \in P_{k-1}$ such that*

$$\|f - Q\|_{L_p[a,b]} \leq c\omega_k \left(f; \frac{b-a}{k}\right)_p,$$

where $c = c(k,p)$.

We will be interested in the case $0 < p < 1$. The case whenever $1 \leq p \leq \infty$ is well known. We do give the proof of $p = \infty$, as well as the case when $k = 1, 0 < p < \infty$, because they are both needed in the proof for $0 < p < 1$. We note here that a simple change of variables shows that it is sufficient to prove the theorem only in the case $[a,b] = [0,1]$. Therefore for the remainder of the paper we are always in $[0,1]$.

Proof in the case $p = \infty$. Assume $f \in L_\infty(0,1)$. We shall make use of the Steklov function, $f_{k,h}$, as an intermediate approximation:

$$f_{k,h}(x) = \frac{1}{h^k} \int_0^h \int_0^h \cdots \int_0^h \sum_{v=1}^{k} (-1)^{v+1} \binom{k}{v} f(x + v(y_1 + \cdots y_k)/k) dy_1 \cdots dy_k.$$

As we have seen before,

$$\|f - f_{k,h}\|_{L_\infty[0,1]} \leq \omega_k(f;h)_\infty, \tag{1}$$

$$\|f_{k,h}^{(k)}\|_{L_\infty[0,1]} \leq c(k) h^{-k} \omega_k(f;h)_\infty. \tag{2}$$

Let $x_0 \in [0,1]$ and set

$$Q(x) = \sum_{v=0}^{k-1} f_{k,h}^{(v)}(x_0) \frac{(x - x_0)^v}{v!},$$

i.e. $Q(x)$ is the (k-1)st order Taylor expansion of $f_{k,h}$ about x_0 and $Q(x) \in P_{k-1}$. Moreover, it can easily be seen using induction on k that

$$f_{k,h}(x) - Q(x) = \frac{1}{(k-1)!} \int_0^{x-x_0} (x - x_0 - s)^{k-1} f_{k,h}^{(k)}(x_0 + s) ds,$$

hence

$$\|f_{k,h} - Q\|_{L_\infty[0,1]} \leq \|f_{k,h}^{(k)}\|_{L_\infty[0,1]}.$$

From this estimate, (1) and (2), setting $h = 1/k$ we get

$$\|f - Q\|_\infty \leq \|f - f_{k,h}\|_\infty + \|f_{k,h} - Q\|_\infty \leq c(k)\omega_k(f, 1/k)_\infty.$$

□

Lemma 2. *Let $f \in L_p[a,b]$, $0 < p < \infty$. Then there exists a constant c such that*

$$\|f - c\|^p_{L_p[a,b]} \le \frac{1}{b-a} \int_a^b \int_a^b |f(x) - f(y)|^p dxdy \tag{3}$$

$$= \frac{2}{b-a} \int_0^{b-a} \int_a^{b-t} |f(x+t) - f(x)|^p dxdt \le 2\omega_1(f; b-a)^p_p, \tag{4}$$

Proof. Consider the function

$$\phi(y) = \int_a^b |f(x) - f(y)|^p dx, \quad y \in [a,b].$$

Then clearly there exists $y_0 \in [a,b]$ such that

$$\phi(y_0) \le \frac{1}{b-a} \int_a^b \phi(y) dy.$$

Therefore, if we set $c = f(y_0)$ we obtain

$$\int_a^b |f(x) - c|^p dx \le \frac{1}{b-a} \int_a^b \int_a^b |f(x) - f(y)|^p dxdy \tag{5}$$

$$= \frac{1}{b-a} \left(\int_a^b \int_a^x |f(x) - f(y)|^p dydx + \int_a^b \int_x^b |f(x) - f(y)|^p dxdy \right)$$

We shall handle each of the integrals above. For the seconde integral, using a substitution $y - x = u$ and then switching the order of integration, we obtain

$$\int_a^b \int_x^b |f(x) - f(y)|^p dydx = \int_a^b \int_0^{b-x} |f(x) - f(u-x)|^p dudx$$

$$= \int_0^{b-a} \int_a^{b-u} |f(u) + f(u+x)|^p dxdu. \tag{6}$$

Similarly for the first integral, with just one extra substituion, we obtain

$$\int_a^b \int_a^x |f(x) - f(y)|^p dydx = \int_a^b \int_0^{x-a} -|f(x) - f(x-u)|^p dudx$$

$$= \int_0^{b-a} \int_{u+a}^b |f(x) - f(x-u)|^p dxdu$$

$$= \int_0^{b-a} \int_a^{b-u} |f(w+u) - f(w)|^p dwdu \tag{7}$$

Combining (5), (6), and (7) completes the proof. □

Lemma 3. *Let $f \in L_p[0,1]$, $0 < p \le 1$. Then for every natural number $n \ge 1$ there exists a step-function φ_n with jumps at the points, $i/n = x_i$, $i = 1, \dots, n-1$, such that*

$$\|f - \varphi_n\|^p_p \le 2n \int_0^{1/n} \int_0^{1-t} |f(x+t) - f(x)|^p dxdt, \quad i = 1, \dots, n.$$

Proof. By lemma 2 there exist constants $c_i, i = 1, \dots, n$ such that

$$\int_{x_{i-1}}^{x_i} |f(x) - c_i|^p dx \le 2n \int_0^{1/n} \int_{x_{i-1}}^{x_i - t} |f(x+t) - f(x)|^p dxdt, i = 1, \dots, n.$$

Then clearly the step function $\varphi_n = c_i$ for $x \in (x_{i-1}, x_i), i = 1, \dots, n$, satisfies the lemma. □

Lemma 4. *Let $f \in L_p[0,1]$, $0 < p < 1$, $k \geq 1$, and $0 \leq \delta \leq \frac{1}{k+1}$. Then we have:*

$$\omega_k(f;\delta)_p^p \leq C\delta^{kp}\left(\int_\delta^{1/(k+1)} t^{-kp}\omega_{k+1}(f;t)_p^p \frac{dt}{t} + \|f\|_p^p\right) \tag{8}$$

Proof. We shall first verify the following identity that we will be using:

$$\Delta_h^k f(x) = 2^{-k}\left(\Delta_{2h}^k f(x) - \sum_{i=0}^{k-1}\sum_{j=i+1}^{k+1} f(x+ih)\right). \tag{9}$$

Recall that we may show by induction that

$$\Delta_{nh}^k f(x) = \sum_{v_1=0}^{n-1}\sum_{v_2=0}^{n-1}\cdots\sum_{v_k=0}^{n-1}\Delta_h^k f(x+(v_1+v_2+\cdots+v_k)h).$$

Hence

$$\begin{aligned}\Delta_{2h}^k f(x) &= \sum_{v_1=0}^{1}\sum_{v_2=0}^{1}\cdots\sum_{v_k=0}^{1}\Delta_h^k f(x+(v_1+v_2+\cdots+v_k)h)\\ &= \Delta_h^k f(x) + \binom{k}{1}\Delta_h^k f(x+h) + \cdots + \binom{k}{k}\Delta_h^k f(x+kh).\end{aligned} \tag{10}$$

We also have the identity,

$$\Delta_h^k f(x+jh) = \Delta_h^k f(x) + \sum_{i=0}^{j-1}\Delta_h^{k+1} f(x+ih). \tag{11}$$

This may be proven directly or by a simple induction arguement on k. Then by (10) and (11)

$$\begin{aligned}\Delta_{2h}^k f(x) &= \sum_{j=0}^{k}\binom{k}{j}\Delta_h^k f(x+jh)\\ &= \sum_{j=0}^{k}\binom{k}{j}\left(\Delta_h^k f(x) + \sum_{i=0}^{j-1}\Delta_h^{k+1} f(x+ih)\right)\\ &= 2^k\Delta_h^k f(x) + \sum_{j=1}^{k}\binom{k}{j}\sum_{i=0}^{j-1}\Delta_h^{k+1} f(x+ih)\\ &= 2^k\Delta_h^k f(x) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\binom{k}{j}\Delta_h^{k+1} f(x+ih)\end{aligned}$$

which implies (9).
We make use of the function,

$$\Omega_k(\delta)_p = \sup_{0<h\leq\delta}\left(\int_0^{1/2}|\Delta_h^k f(x)|^p dx\right)^{1/p}.$$

By (9) and since $0 < p < 1$, we get

$$|\Delta_h^k f(x)|^p \leq 2^{-kp}\left(|\Delta_{2h}^k f(x)|^p + \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\binom{k}{j}^p|\Delta_h^{k+1} f(x+ih)|^p\right).$$

Now for $0 < h < \delta \leq \frac{1}{4k}$,$0 \leq x \leq 1/2$, and $i \leq k-1$ we have $x + ih \leq 3/4$, thus we obtain

$$\begin{aligned}\Omega_k(\delta)_p^p &\leq \sup_{0<h\leq\delta} 2^{-kp}\int_0^{1/2} |\Delta_{2h}^k f(x)|^p + \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\binom{k}{j}^p 2^{-kp}\sup_{0<h\leq\delta}\int_0^{1/2}|\Delta_h^{k+1}f(x+ih)|^p dx \\ &\leq 2^{-kp}\Omega_k(2\delta)_p^p + C\omega_{k+1}(f;\delta)_p^p, \end{aligned} \tag{12}$$

where $C = C(k,p)$. Let $r \geq 1$ and $0 < \delta \leq 1/(2^{r+1}k)$. Then by (12) we have for $i = 0, 1, \ldots, r-1$

$$2^{-kpi}\Omega_k(2^i\delta)_p^p \leq 2^{-kp(i+1)}\Omega_k(2^{i+1}\delta)_p^p + C2^{-kpi}\omega_{k+1}(f, 2^i\delta)_p^p.$$

Summing the inequalities over $i = 0, 1, \ldots, r-1$ and cancling terms seen on both the right side and left side of the inequality we get

$$\begin{aligned}\Omega_k(\delta)_p^p &\leq 2^{-kpr}\Omega_k(2^r\delta)_p^p + C\sum_{i=0}^{r-1}2^{-kpi}\omega_{k+1}(f;t)_p^p \\ &\leq 2^{-kp(r-1)}\|f\|_p^p + C_1\delta^{kp}\sum_{i=1}^{r-1}\int_{2^i\delta}^{2^{i+1}\delta} t^{-kp}\omega_{k+1}(f;t)_p^p\frac{dt}{t} \\ &= 2^{-kp(r-1)}\|f\|_p^p + C_1\delta^{kp}\int_\delta^{2^r\delta} t^{-kp}\omega_{k+1}(f;t)_p^p\frac{dt}{t}\end{aligned}$$

The second inequality above follows from the simple inequality $\Omega_k(\delta)_p \leq 2^k\|f\|_p$, for $\delta k \leq 1/2$. From here simply note that $\omega_k(f;\delta)_p^p \leq 2\Omega_k(\delta)_p^p$, hence

$$\omega_k(f;\delta))p^p \leq 2\cdot 2^{-kp(r-1)}\|f\|_p^p + C_1\delta^{kp}\int_\delta^{2^r\delta} t^{-kp}\omega_{k+1}(f;t)_p^p\frac{dt}{t} \tag{13}$$

Let $0 < \delta \leq 1/4k$. Choose $r \geq 1$ such that $1/2^{r+2}k < \delta \leq 1/2^{r+1}k$. Then by (13) we obtain

$$\omega_k(f;\delta)_p^p \leq C\delta^{kp}\left(\int_\delta^{1/4k} t^{-kp}\omega_{k+1}(f;t)_p^p\frac{dt}{t} + \|f\|_p^p\right)$$

which gives the lemma ☐

Corollary 5 (Marchaud)**.** *Let $f \in L_p[0,1]$, $0 < p < 1$, $m > k \geq 1$, and $0 < \delta \leq 1$. Then we have*

$$\omega_k(f;\delta)_p^p \leq C\delta^{kp}\left(\int_\delta^1 t^{-kp}\omega_m(f;t)_p^p\frac{dt}{t} + \|f\|_p^p\right).$$

Proof. We shall prove it by induction with respect to m. The inequality holds for $m = k+1$ by the previous lemma.
Suppose that it holds for some $m \geq k+1$. Then we obtain,

$$\begin{aligned}\omega_k(f,\delta)_p^p &\leq C\delta^{kp}\left(\int_\delta^1 t^{-kp-1}\omega_m(f,t)_p^p dt + \|f\|_p^p\right) \\ &\leq C\delta^{kp}\left(\int_\delta^1 t^{-kp-1}\left[C\delta^{kp}\int_t^1 u^{-mp-1}\omega_{m+1}(f;t)_p^p du + \|f\|_p^p\right] + \|f\|_p^p\right) \qquad (14) \\ &\leq C_1\delta^{kp}\left(\int_\delta^1 t^{-kp}\omega_{m+1}(f;t)_p^p\frac{dt}{t} + \|f\|_p^p\right), \qquad (15)\end{aligned}$$

where in (14) we applied lemma 4 and in (15) we apply the Hardy inequality. ☐

Proof of theorem 1 in the case $0 < p \le 1$. Suppose the theorem does not hold. Then there exists a sequence of functions $\{f_m\}_{m=1}^{\infty}, f_m \in L_p(0,1)$, such that

$$\inf_{Q \in P_{k-1}} \|f_m - Q\|_p^p > m\omega_k(f_m; 1/k)_p^p, \quad m = 1, 2, \dots$$

Since the set of all polynomials $Q \in P_{k-1}$ such that $\|Q\|_p \le 1$ is a compact set in $L_p(0,1)$, then for each m there exists a polynomial $Q_m \in P_{k-1}$ such that

$$\|f_m - Q_m\|_p = \inf_{Q \in P_{k-1}} \|f_m - Q\|_p. \tag{16}$$

Consequently

$$\|f_m - Q_m\|_p^p > m\omega_k(f_m; 1/k)_p^p, \quad m = 1, 2, \dots$$

We will set

$$g_m = \frac{f_m - Q_m}{\|f_m - Q_m\|_p}$$

Then it is clear that

$$\inf_{Q \in P_{k-1}} \|g_m - Q\|_p = \|g_m\|_p = 1 \tag{17}$$

and by (16) we have

$$\omega_k(g_m; k^{-1})_p^p < 1/m, \quad m = 1, 2, \dots \tag{18}$$

We shall prove that $\{g_m\}_1^{\infty}$ is a precompact set in $L_p(0,1)$, i.e. there exists a function $g \in L_p$ and a subsequence $\{g_{m_i}\}_1^{\infty}$ such that $\|g - g_{m_i}\|_p \to 0$ as $i \to \infty$. To this end it is sufficient to prove that for all $\epsilon > 0$ there exists a finite ϵ-net for $\{g_m\}_1^{\infty}$ in $L_p(0,1)$.

It follows from corollary 5 with $k = 1, m = k$ and (17), (18) that

$$\omega_1(g_m; \delta)_p^p \le c\delta^p \left(\int_\delta^1 t^{-p} \frac{1}{m} \frac{dt}{t} + 1 \right) \le c_1 \left(\frac{1}{m} + \delta^p \right)$$

for $0 \le \delta \le 1$ and $m = 1, 2, \dots$. Therefore if follows that for each $\epsilon > 0$ there exists $m_0 > 0$ and $\delta_0 > 0$ such that

$$\omega_1(g_m; \delta)_p^p \le \epsilon \quad \text{for } 0 < \delta < \delta_0 \quad \text{and } m \ge m_0. \tag{19}$$

Fix $n > 1/\delta_0$. Then by lemma 3 and (19) that for each $m > m_0$ there exists a step function $\varphi_{m,n}$ with points of discontinuity $i/n, i = 1, \dots, n-1$, such that

$$\|g_m - \varphi_{m,n}\|_p^p \le 2\omega_1(g_m; n^{-1})_p^p < 2\epsilon. \tag{20}$$

On the other hand by (17) and (20) we get

$$\|\varphi_{m,n}\|_p^p \le \|g_m\|_p^p + \|g_m - \varphi_{m,n}\|_p^p < 1 + 2\epsilon.$$

Since $\varphi_{m,n}(x)$ is constant for $x \in ((i-1)/n, i/n), i = 1, \dots, n$, for $m > m_0$ we have

$$\begin{aligned}
\|\varphi_{m,n}\|_\infty &\le \sum_{i=1}^n \left| \varphi_{m,n}\left(\frac{i - .5}{n} \right) \right| n^{1/p} (1/n)^{1/p} \\
&= \sum_{i=1}^n \left(n \int_{(i-1)/n}^{i/n} |\varphi_{m,n}(x)|^p dx \right)^{1/p} \\
&\le \left(\sum_{i=1}^n n \int_{(i-1)/n}^{i/n} |\varphi_{m,n}|^p dx \right)^{1/p} \\
&= \left(n \int_0^1 |\varphi_{m,n}|^p dx \right)^{1/p} \le ((1 + 2\epsilon)n)^{1/p} = M.
\end{aligned}$$

Now consider the set Ψ of all step functions ϕ of the type

$$\phi(x) = r\epsilon^{1/p}, \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right), \quad i = 1, \dots, n, \quad r = 0, \pm 1, \dots, \quad \|\phi\|_\infty \le M.$$

Clearly

$$\inf_{\phi \in \Psi} \|\phi_{m,n} - \phi\|_p^p \le \int_0^1 (\epsilon^{1/p})^p = \epsilon$$

and therefore Ψ is an ϵ-net for the set $\{\phi_{m,n}\}_{m=m_0+1}^\infty$. From this and (20) it follows that

$$\inf_{\phi \in \Psi} \|g_m - \phi\|_p^p \le \|g_m - \phi_{m,n}\|_p^p + \inf_{\phi \in \Psi} \|\phi_{m,n} - \phi\|_p^p \le 3\epsilon,$$

hence Ψ is a 3ϵ-net for $\{g_m\}_{m=m_0+1}^\infty$. Thus $\{g_m\}_{m=m_0+1}^\infty$ is a precompact set in $L_p(0,1)$. So for an appropriate subsequence $\{g_{m_i}\}_{i=1}^\infty$ there exists $g \in L_p(0,1)$ such that $\|g_{m_i} - g\|_p \to 0$ as $i \to \infty$. Hence, in view of (17) we have

$$\inf_{Q \in P_{k-1}} \|g - Q\|_p^p \ge \inf_{Q \in P_{k-1}} \|g_{m_i} - Q\|_p^p - \|g - g_{m_i}\|_p^p \to 1 \quad \text{as} \quad i \to \infty$$

and

$$\inf_{Q \in P_{k-1}} \|g - Q\|_p^p \le \inf_{Q \in P_{k-1}} \|g_{m_i} - Q\|_p^p + \|g - g_{m_i}\|_p^p \to 1 \quad \text{as} \quad i \to \infty.$$

Therefore

$$\inf_{Q \in P_{k-1}} \|g - Q\|_p^p = 1. \tag{21}$$

On the other hand by (18) we get

$$\begin{aligned} \omega_k(g; k^{-1})_p^p &\le \omega_k(g_{m_i}; k^{-1})_p^p + \omega_k(g - g_{m_i}; k^{-1})_p^p \\ &\le \omega_k(g_{m_i}; k^{-1})_p^p + 2^{kp}\|g - g_{m_i}\|_p^p \to 0 \quad \text{as} \quad i \to \infty. \end{aligned}$$

Thus $\omega_k(g; k^{-1})_p = 0$. As we shall show below this equality implies that $g \equiv Q$ a.e. for some $Q \in P_{k-1}$, which contradicts (). □

Lemma 6. *Let $f \in L_p(0,1)$, $0 < p \le 1$, and $\omega_k(f; k^{-1})_p = 0$. Then there exists a polynomial $Q \in P_{k-1}$ such that $f \equiv Q$ almost everywhere in $[0,1]$.*

Proof. We shall prove the lemma by induction with respect to k. In the case $k = 1$, the lemma follows by lemma 2. Now suppose the lemma holds for some $k \ge 1$. Suppose that

$$\omega_{k+1}(f; (k+1)^{-1})_p^p = \sup_{0 \le h \le (k+1)^{-1}} \int_0^{1-(k+1)h} |\Delta_h^{k+1} f(x)|^p dx = 0. \tag{22}$$

First we shall prove that

$$\int_0^{1-kh_1-h} |\Delta_{h_1}^k \Delta_h^1 f(x)|^p dx = 0, \quad h_1, h \ge 0, \quad kh_1 + h \le 1. \tag{23}$$

If $h_1 = \alpha h$ and $\alpha = m/n$ for some $m, n \in \mathbb{Z}^+$, then appling the identity from lemma 4 twice we obtain

$$\begin{aligned} |\Delta_{(m/n)h}^k \Delta_h^1 f(x)|^p &\le \sum_{v_1=0}^{m-1} \cdots \sum_{v_k=0}^{m-1} \left| \Delta_{h/n}^k \Delta_{\frac{hn}{n}}^1 f\left(x + \frac{h}{n}(v_1 + \cdots + v_k)\right) \right|^p \\ &= \sum_{v_1=0}^{m-1} \cdots \sum_{v_k=0}^{m-1} \left| \Delta_{h/n}^k \sum_{v=0}^{n-1} \Delta_{h/n}^1 f\left(x + \frac{h}{n}(v_1 + \cdots + v_k + v)\right) \right|^p \\ &\le \sum_{v_1=0}^{m-1} \cdots \sum_{v_k=0}^{m-1} \sum_{v=0}^{n-1} \left| \Delta_{h/n}^{k+1} f\left(x + \frac{h}{n}(v_1 + \cdots + v_k + v)\right) \right|^p. \end{aligned}$$

Integrating with respect to $x \in [0, 1-(km/n+1)+h]$ and using (22) we conclude that (23) holds true.
No we need to show it holds true for irrational numbers α. So suppose that $h_1 = \alpha h$, $a > 0$ and irrational number. Choose a sequence $\{a_i\}_{i=1}^{\infty}$ of rational numbers such that $\alpha_i \to \alpha$ and $0 < \alpha_i < \alpha$. We have

$$\begin{aligned}
|\Delta_{\alpha h}^k \Delta_h^1 f(x)| &\le |\Delta_{\alpha_i h}^k \Delta_h^1 f(x)| + |\Delta_{\alpha h}^k \Delta_h^1 f(x) - \Delta_{\alpha_i h}^k \Delta_h^1 f(x)| \\
&= |\Delta_{\alpha_i h}^k \Delta_h^1 f(x)| + \left| \sum_{v=0}^{k} \binom{k}{v} \Delta_h^1 \left(f(x+v\alpha h) - f(x+v\alpha_i h) \right) \right| \\
&\le |\Delta_{\alpha_i h}^k \Delta_h^1 f(x)| + \sum_{v=0}^{k} \binom{k}{v} (|f(x+v\alpha h+h) - f(x+v\alpha_i h+h)| \\
&+ |f(x+v\alpha h) - f(x+v\alpha_i h)|) \\
&= |\Delta_{\alpha_i h}^k \Delta_h^1 f(x)| + \sum_{v=0}^{k} \binom{k}{v} |\Delta_{v(\alpha-\alpha_i)h}^1 f(x+h)| + |\Delta_{v(\alpha-\alpha_i)h}^1 f(x)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{1-k\alpha h-h} |\Delta_{\alpha_i h}^k \Delta_h^1 f(x)| &\le \int_0^{1-k\alpha_i h-h} |\Delta_{\alpha h}^k \Delta_h^1 f(x)| + c(k,p)\omega_1(f; k(\alpha-\alpha_i)h)_p^p \\
&= c(k,p)\omega_1(f; k(\alpha-\alpha_i)h)_p^p, \qquad (24)
\end{aligned}$$

where we have used that (23) holds for α_i a rational number. Since $\omega_1(f;\delta)_p \to 0$ as $\delta \to 0$ (24) implies (23). □

Now since (23) holds, we have $g(x) = \Delta_h^1 f(x)$ is such that

$$\int_0^{1-kh_1-h} |\Delta_{h_1}^k g(x)|^p = 0, \quad h_1, h \ge 0, \quad kh_1 + h \le 1,$$

hence in view of our induction hypothesis, $g(x)$ is equal to a polynomial of degree $k-1$ a.e. for $x \in [0, 1-h]$. There for there exists a polynomial $Q_h \in P_{k-1}$ such that

$$f(x+h) - f(x) = \sum_{v=0}^{k-1} a_v(h) x^v \qquad (25)$$

almost everywhere in $[0, 1-h]$.
We shall prove that each coefficient $a_v(h)$ is a continuous function of $h \in [0,1)$. Let $0 \le h_1 \le h_2 < 1$. Then we note that

$$f(x+h_1) - f(x+h_2) = \sum_{v=0}^{k-1} (a_v(h_1) - a_v(h_2)) x^v. \qquad (26)$$

We note that for $P(x) \in P_{k-1}[0,b], b > 0$, where $P(x) = \sum_{i=0}^{k-1} c_v x^v$, we have the following as norms:

$$\|P\|_1 := \sum_{v=0}^{k-1} |c_v||b|^v$$

$$\|P\|_2 := \left(\frac{1}{b} \int_0^b |P(x)|^p dx \right)^{1/p}.$$

Since all norms on a finite dimensional space are equivalent, we may apply this to (26) to obtain

$$\sum_{v=0}^{k-1} |a_v(h_1) - a_v(h_2)||1-h|^v \leq c \left(\frac{1}{1-h_2} \int_0^{1-h_2} |f(x+h_1) - f(x+h_2)|^p dx \right)^{1/p}$$
$$\leq c_1 \left(\frac{1}{1-h_2} \omega_1(f; |h_1 - h_2|)_p^p \right).$$

Since $\omega_1(f;\delta)_p \to 0$, it follows that $a_v(h)$ is a continuous function of $h \in [0,1)$.
Applying an arbitrary $(k+1)$th difference Δ_t^{k+1} to (25) as a function of h we obtain

$$\Delta_t^{k+1} f(x+h) = \sum_{v=0}^{k-1} (\Delta_t^{k+1} a_v(h)) x^v$$

for almost all $x \in (0, 1-h-(k+1)t)$ and $t, h \geq 0, h+(k+1)t < 1$. By our initial assumptions, for almost all $x \in (0, 1-h-(k+1)t), h+(k+1)t < 1$, we have $\Delta_h^{k+1} f(x) = 0$, and since $a_v(h)$ is a continuouse function of h we have

$$\Delta_t^{k+1} a_v(h) = 0, \quad 0 \leq h < 1-(k+1)t, \quad 0 \leq t < 1/(k+1), \quad v = 0, \ldots, k-1.$$

Since we have already proven Whitney's theorem for $p = \infty$, we conclude that $a_v(h)$ coincides with some polynomial $Q(h) \in P_k$ for all $h \in [0,1), v = 0, \ldots, k-1$. In view of (25), we conclude that $f(x+h)$ coincides with a polynomial in P_k as a function of h. This gives the lemma.